

1-factor- and cycle covers of cubic graphs

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Abstract

For $k \geq 1$, let $m_k(G)$ be the maximum number of edges of a bridgeless cubic graph G that can be covered by the union of k 1-factors of G , and $\mu_k(G) = |E(G)| - m_k(G)$. The main focus of the paper are covers with three 1-factors. It introduces the core of a cubic graph and studies the structure of cores. It shows that if $\mu_3(G) \neq 0$, then $2\mu_3(G)$ is an upper bound for the girth of G . A bridgeless cubic graph G without non-trivial 3-edge-cut and $\mu_3(G) \leq 4$ has a Fulkerson coloring. If G is bridgeless and $\mu_3(G) \leq 6$ or $\mu_3(G) < \text{girth}(G)$, then G has three 1-factors with empty intersection. It further proves some new upper bounds for the length of cycle covers of bridgeless cubic graphs. Cubic graphs with $\mu_4(G) = 0$ have a 4-cycle cover of length $\frac{4}{3}|E(G)|$ and a 5-cycle double cover. These graphs also satisfy two conjectures of Zhang [19], who conjectured that every bridgeless graph has a shortest 4-cycle cover, and that every 3-connected graph has a shortest cycle cover such that every edge is contained in at most two cycles. We also give a negative answer to a problem of Zhang [19].

1 Introduction

The terms graphs and multigraphs will be used in this paper. Graphs may contain multiple edges but no loops, while multigraphs may have multiple edges or loops. A 1-factor of a multigraph G is a spanning 1-regular subgraph of G . One of the first and famous theorems in graph theory, Petersen's Theorem from 1891, states that every bridgeless cubic graph has a 1-factor. Indeed, for every edge e of a bridgeless cubic graph G there is a 1-factor that contains e .

Let G be a cubic graph and $k \geq 1$. We define $m_k(G) = \max\{|\bigcup_{i=1}^k M_i| : M_1, \dots, M_k \text{ are 1-factors of } G\}$, and $\mu_k(G) = |E(G)| - m_k(G)$. A k -cover of G is the union of k 1-factors of G .

If $\mu_3(G) = 0$, then G is 3-edge-colorable. Cubic graphs with chromatic index 4 are subject of many papers since these graphs are potential counterexamples

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to many hard conjectures, see e.g. [3]. Cyclically 4-edge-connected cubic graphs with chromatic index 4 and girth at least 5 are also called *snarks*.

If $\mu_5(G) = 0$, then the edges of G can be covered by five 1-factors. This is conjectured to be true for all bridgeless cubic graphs by Berge. In [16] it is shown that Berge's conjecture is equivalent to the celebrated Fulkerson conjecture, which states that every bridgeless cubic graph has six 1-factors such that every edge is contained in precisely two of them. We say that G has a Fulkerson coloring if $\mu_5(G) = 0$.

Conjecture 1.1 (Berge-Fulkerson Conjecture [7]) *Let G be a cubic graph. If G is bridgeless, then $\mu_5(G) = 0$.*

The following conjecture of Fan and Raspaud is true if the Berge-Fulkerson Conjecture is true.

Conjecture 1.2 ([5]) *Every bridgeless cubic graph has three 1-factors M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

Eulerian graphs will be called *cycles* in the following. A *cycle cover* of a graph G is a set \mathcal{C} of cycles such that every edge of G is contained in at least one cycle. It is a *double cycle cover* if every edge is contained in precisely two cycles, and a k -(double) cycle cover ($k \geq 1$) if \mathcal{C} consists of at most k cycles. Circuits of length 2 are allowed, but the two edges of such a circuit must be different. The following conjecture was stated by Celmins and Preissmann independently.

Conjecture 1.3 (see [19]) *Every bridgeless graph has a 5-cycle double cover.*

This conjecture is equivalent to its restriction to cubic graphs. In this case a cycle is a 2-regular graph. If it is connected, then we call it *circuit*; that is, a cycle is a set of disjoint circuits. A cycle is *even* if it consists of even circuits. The length of a cycle cover \mathcal{C} is the number of edges of \mathcal{C} . A 3-edge-colorable cubic graph G has a 2-cycle cover of length $\frac{4}{3}|E(G)|$. Alon and Tarsi stated the following conjecture.

Conjecture 1.4 ([1]) *Every bridgeless graph G has a cycle cover of length at most $\frac{7}{5}|E(G)|$.*

Section 2 studies 3-covers of cubic graphs. We introduce the core of a cubic graph and prove some properties of cores. Using structural properties of cores we show that if $\mu_3(G) \neq 0$, then $2\mu_3(G)$ is an upper bound for the girth of a cubic graph G . If G is a bridgeless cubic graph without non-trivial 3-edge-cut and $\mu_3(G) \leq 4$, then G has a Fulkerson coloring. If G is a bridgeless cubic graph and $\mu_3(G) \leq 6$, then G satisfies Conjecture 1.2. For $\mu_3(G) < \text{girth}(G)$ we prove the stronger result, that any three 1-factors whose union covers all but $\mu_3(G)$

edges of G have an empty intersection. We prove some new upper bounds for the length of a shortest cycle covers of bridgeless cubic graphs. If G is trianglefree and $\mu_3(G) \leq 5$, then G has an even 3-cycle cover of length at most $\frac{4}{3}|E(G)| + 2$. Furthermore, if $\mu_3(G) \leq \text{girth}(G)$, then G has an even cycle cover of length at most $\frac{52}{35}|E(G)|$. With those methods, some earlier results of [5] and [13] are improved to even 3-cycle covers.

Section 3 proves that if a cubic graph G has four 1-factors $M_1 \dots, M_4$ such that $\bigcap_{i=1}^4 M_i = \emptyset$ and $|E(G) - \bigcup_{i=1}^4 M_i| = k$, then it has an even 4-cycle cover of length $\frac{4}{3}|E(G)| + 4k$. Furthermore, if $\mu_4(G) \in \{0, 1, 2, 3\}$, then G has an even 4-cycle cover of length $\frac{4}{3}|E(G)| + 4\mu_4(G)$. If $\mu_4(G) = 0$, then G has an even 4-cycle cover of length $\frac{4}{3}|E(G)|$, and a 5-cycle double cover. Furthermore, these graphs have a shortest 4-cycle cover with edge depth at most 2. Zhang [19] conjectured that every bridgeless graph has a shortest 4-cycle cover (Conjecture 8.11.5) and that every 3-connected graph has a shortest cycle cover with edge depth at most 2 (Conjecture 8.11.6). We further give a negative answer to Problem 8.11.4 of [19]. It seems that snarks which have only cycle covers with more than $\frac{4}{3}|E(G)|$ edges are rare. So far only two such graphs with at most 36 vertices are known [4]. One of these graphs is the Petersen graph and the other one has 34 vertices. Both graphs have a cycle cover of length $\frac{4}{3}|E(G)| + 1$.

The paper closes with a few remarks on hypohamiltonian snarks.

2 3-covers and the core of a cubic graph

Let M be a multigraph. If $X \subseteq E(M)$, then $M[X]$ denotes the graph whose vertex set consists of all vertices of edges of X and whose edge set is X .

Let G be a cubic graph that has three 1-factors M_1, M_2 and M_3 . The set of the edges which are in more than one 1-factor is denoted by \mathcal{M} , and the set of edges which are not contained in any of the three 1-factors by \mathcal{U} ; that is, $\mathcal{M} = \bigcup_{i \neq j} (M_i \cap M_j)$ and $\mathcal{U} = E(G) - \bigcup_{i=1}^3 M_i$. Clearly, \mathcal{M} and \mathcal{U} are disjoint. We consider the graph $G[\mathcal{M} \cup \mathcal{U}]$ which is induced by the union of \mathcal{M} and \mathcal{U} . If $M_1 = M_2 = M_3$, then $G = G[\mathcal{M} \cup \mathcal{U}]$. Hence we ask the three 1-factors to be pairwise different in the following definition.

Let M_1, M_2 and M_3 be three pairwise different 1-factors of a cubic graph G , $k \geq 0$, and $|E(G) - \bigcup_{i=1}^3 M_i| = k$. The k -core of G with respect to M_1, M_2 , and M_3 is the subgraph G_c of G which is induced by $\mathcal{M} \cup \mathcal{U}$; that is, $G_c = G[\mathcal{M} \cup \mathcal{U}]$. If the value of k is irrelevant, then we say that G_c is a core of G .

It is a folklore result that for each edge e of a bridgeless cubic graph there is a 1-factor containing e . Hence we easily get the following proposition.

Proposition 2.1 *Let G be a cubic graph. If G has a 1-factor, then it has a core. Furthermore, if G is bridgeless, then for every $v \in V(G)$ there is a core G_c such that $v \notin V(G_c)$.*

Theorem 2.2 *Let G_c be a core of a cubic graph G with respect to three 1-factors M_1 , M_2 and M_3 , and let K_c be a component of G_c . Then \mathcal{M} is a 1-factor of G_c , and*

- 1) K_c is either an even circuit or it is a subdivision of a cubic multigraph K , and
- 2) if K_c is a subdivision of a cubic multigraph K , then $E(K_c) \cap \bigcap_{i=1}^3 M_i$ is a 1-factor of K .

Proof. Let G_c be the core of G with respect to M_1 , M_2 and M_3 . The set \mathcal{M} is a matching in G . Hence every vertex $v \in V(G_c)$ is incident to at most one edge of \mathcal{M} . On the other hand, since M_1 , M_2 and M_3 are 1-factors, v cannot be incident to three edges of \mathcal{U} . Hence \mathcal{M} is a 1-factor of G_c .

- 1) Indeed, if v is incident to an edge of $\bigcap_{i=1}^3 M_i$, then $d_{G_c}(v) = 3$, and if it is incident to an edge of $(M_l \cap M_j) - \bigcap_{i=1}^3 M_i$ ($l \neq j$), then $d_{G_c}(v) = 2$. Let K_c be a component of G_c . If it has no trivalent vertices, then $E(K_c) \cap \mathcal{M}$ is a 1-factor of K_c and hence K_c is an even circuit, whose edges are in \mathcal{M} and in \mathcal{U} , alternately. If K_c contains trivalent vertices, then it is a subdivision of a cubic multigraph K .
- 2) If $E(K_c) \cap \bigcap_{i=1}^3 M_i \neq \emptyset$, then the vertices which are incident to an edge of $\bigcap_{i=1}^3 M_i$ have degree 3 in G_c . Hence $E(K_c) \cap \bigcap_{i=1}^3 M_i$ is a matching in K_c that covers all trivalent vertices of K_c . Since K is obtained from K_c by suppressing the bivalent vertices and the edges of $E(K_c) \cap \bigcap_{i=1}^3 M_i$ are unchanged, it follows that $E(K_c) \cap \bigcap_{i=1}^3 M_i$ is a 1-factor of K . \square

In [6] it is proved that if G has three 1-factors M_1 , M_2 , M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$, then $\mathcal{M} \cup \mathcal{U}$ induces even circuits in G . The next theorem summarizes some properties of cores.

Theorem 2.3 *Let $k \geq 0$, and G_c be a k -core of a cubic graph G with respect to three 1-factors M_1 , M_2 and M_3 . Then,*

- 1) if $k < 3$, then G is 3-edge-colorable.
- 2) $|V(G_c)| = 2k - 2|\bigcap_{i=1}^3 M_i|$, and $|E(G_c)| = 2k - |\bigcap_{i=1}^3 M_i|$.
- 3) $\text{girth}(G_c) \leq 2k$.
- 4) G_c has at most $2k/\text{girth}(G_c)$ components.

Proof. 1) If $k < 3$, then there are i, j such that $1 \leq i < j \leq 3$ and $M_i \cap M_j = \emptyset$. Hence $M_i \cup M_j$ is an even 2-factor of G and therefore G is 3-edge-colorable.

Claim 2.3.1 $|\mathcal{M}| = k - |\bigcap_{i=1}^3 M_i|$

Proof. For $i \neq j$, let $(M_i \cap M_j)_2 = (M_i \cap M_j) - \bigcap_{i=1}^3 M_i$. Every vertex of an edge of $(M_i \cap M_j)_2$ has degree 1 in $G[\mathcal{U}]$, and every vertex of an edge of $\bigcap_{i=1}^3 M_i$ has degree 2 in $G[\mathcal{U}]$. Hence $k = \frac{1}{2}[2(|(M_1 \cap M_2)_2| + |(M_1 \cap M_3)_2| + |(M_2 \cap M_3)_2|) + 4|\bigcap_{i=1}^3 M_i|] = |(M_1 \cap M_2)_2| + |(M_1 \cap M_3)_2| + |(M_2 \cap M_3)_2| + 2|\bigcap_{i=1}^3 M_i|$ and therefore, $|\mathcal{M}| = |(M_1 \cap M_2)_2| + |(M_1 \cap M_3)_2| + |(M_2 \cap M_3)_2| + |\bigcap_{i=1}^3 M_i| = k - |\bigcap_{i=1}^3 M_i|$. \square

Since \mathcal{M} is a 1-factor of G_c , it follows that $|V(G_c)| = 2k - 2|\bigcap_{i=1}^3 M_i|$. Furthermore, $|\mathcal{U}| = k$, $\mathcal{M} \cap \mathcal{U} = \emptyset$, and $E(G_c) = \mathcal{M} \cup \mathcal{U}$, imply that $|E(G_c)| = 2k - |\bigcap_{i=1}^3 M_i|$.

3) Since G_c has minimum degree 2 and at most $2k$ vertices, it follows that it contains a circuit of length at most $2k$.

4) Every component is 2-edge-connected and therefore it contains a circuit. Since \mathcal{M} is a 1-factor of G_c , it follows that each circuit contains at least $\frac{1}{2}\text{girth}(G_c)$ edges of \mathcal{U} . Hence there are at most $\frac{2k}{\text{girth}(G_c)}$ pairwise disjoint circuits in G_c . \square

Corollary 2.4 *Let G be a cubic graph. If $\mu_3(G) \neq 0$, then $\mu_3(G) \geq 3$ and $\text{girth}(G) \leq 2\mu_3(G)$.*

Proof. If $\mu_3(G) > 0$, then G has chromatic index 4. Hence the result follows with Theorem 2.3. \square

With view on Corollary 2.4 it would be interesting to know for which integers k there is a bridgeless cubic graph G (a snark G) with $\mu_3(G) = k$.

2.1 The conjecture of Fan and Raspaud

If a core G_c of a cubic graph is a cycle, then we say that G_c is a *cyclic* core. A cubic graph G has three 1-factors M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$ if and only if G has a cyclic core. Hence, Conjecture 1.2 can be formulated as a conjecture on cores in bridgeless cubic graphs.

Conjecture 2.5 (Conjecture 1.2) *Every bridgeless cubic graph has a cyclic core.*

Let K_2^3 be the unique cubic graph on two vertices which are connected by three edges.

Theorem 2.6 *Let $k \geq 0$, and G be a cubic graph with $\mu_3(G) = k$. If $\text{girth}(G) \geq k$ ($> k$), then every k -core of G is bipartite (cyclic).*

Proof. Let G_c be an arbitrary k -core of G with respect to three 1-factors M_1, M_2, M_3 , and $\text{girth}(G) \geq k$. Clearly, $\text{girth}(G_c) \geq \text{girth}(G)$, and note that a cyclic core is bipartite. By Lemma 2.2, G_c contains a circuit C , and \mathcal{M} is a 1-factor of G_c . By Lemma 2.3, $|E(G_c)| = 2k - |\bigcap_{i=1}^3 M_i|$ and hence $|E(G_c)| \leq 2\text{girth}(G) - |\bigcap_{i=1}^3 M_i|$. This implies that if G_c has two disjoint circuits, then $|\bigcap_{i=1}^3 M_i| = 0$, and hence G_c is cyclic in this case.

Thus, we may assume that any two circuits in G_c intersect. Hence G_c is 2-edge-connected. Suppose that there is an edge e in $\bigcap_{i=1}^3 M_i$. Since $G_c - \bigcap_{i=1}^3 M_i$ is a circuit C_c , it follows that e is a chord of C_c . Thus, there are two circuits C_1 and C_2 with $E(C_1) \cap E(C_2) = \{e\}$ and $E(C_1) \cap E(C_2) \cap \mathcal{U} = \emptyset$.

If $\text{girth}(G_c) > k$, then each of C_1 and C_2 contains more than $\frac{k}{2}$ edges of \mathcal{U} . This contradicts $|\mathcal{U}| = k$, and hence G_c is cyclic in this case.

It remains to consider the case when $\text{girth}(G) = k$. Then each of the two circuits C_1 and C_2 contains precisely $\frac{k}{2}$ edges of \mathcal{U} . Hence k is even. Furthermore $2k - |\bigcap_{i=1}^3 M_i| = |E(G_c)| \geq |E(C_1) \cup E(C_2) \cup \{e\}| \geq 2\text{girth}(G) - 2 + 1 = 2k - 1$. Hence $\bigcap_{i=1}^3 M_i = \{e\}$, and G_c is a subdivision of K_2^3 . Since every circuit of G_c is even, it follows that G_c is bipartite. \square

(*) Remark to Theorem 2.6 : Note that we proved the following stronger statement in the last case: Let G_c be a $\mu_3(G)$ -core of a cubic graph G . If $\text{girth}(G_c) = \mu_3(G)$ and $\bigcap_{i=1}^3 M_i \neq \emptyset$, then $\mu_3(G)$ is even and G_c is a subdivision of K_2^3 .

Let G be a bridgeless cubic graph. The minimum number of odd circuits in a 2-factor of G is the oddness of G , which is denoted by $\omega(G)$. Máčajová and Škoviera proved that Conjecture 1.2 is true for bridgeless cubic graphs with oddness at most 2.

Theorem 2.7 ([14]) *Let G be a bridgeless cubic graph. If $\omega(G) \leq 2$, then G has a cyclic core.*

For the proof of the next theorem we will use the following proposition.

Proposition 2.8 ([18]) *Let G be a bridgeless non-3-edge-colorable cubic graph. There is a proper 4-edge-coloring of G with a color class that contains precisely two edges if and only if $\omega(G) = 2$.*

Theorem 2.9 *Let G be a simple bridgeless cubic graph. If $\mu_3(G) \leq 6$, then G has a cyclic core. In particular, if G is trianglefree and $\mu_3(G) \leq 5$, then every $\mu_3(G)$ -core is cyclic.*

Proof. Let G_c be a core of G with respect to M_1, M_2, M_3 , and $|E(G) - \bigcup_{i=1}^3 M_i| = \mu_3(G)$. If $\mu_3(G) = 0$, then there is nothing to prove. If $\text{girth}(G) > \mu_3(G)$, then the statement follows with Theorem 2.6.

Let $\text{girth}(G) \leq \mu_3(G)$ and suppose to the contrary that there is an edge e in $\bigcap_{i=1}^3 M_i$. Thus $\mu_3(G) \in \{4, 5, 6\}$.

We first consider the case when $\text{girth}(G) = \mu_3(G)$. If $\text{girth}(G_c) > \text{girth}(G)$, then the result follows as in the proof of Theorem 2.6. Thus we assume that $\text{girth}(G_c) = \mu_3(G)$. Then by Remark (*), $\mu_3(G)$ is even. Hence $\mu_3(G) \in \{4, 6\}$. Furthermore G_c contains two circuits C_1 and C_2 , and one of them is of length 4, which leads to a contradiction if $\mu_3(G) = 6$. It remains to consider the case when $\mu_3(G) = 4$. Then C_1 and C_2 both have length 4, and $E(C_1) \cap E(C_2) = \{e\}$.

(**) For $i \neq j$, let $(M_i \cap M_j)_2 = (M_i \cap M_j) - \bigcap_{i=1}^3 M_i$. Then there are i, j , such that $1 \leq i < j \leq 3$ and $(M_i \cap M_j)_2 \cap E(C_1) = \emptyset$, say $(M_1 \cap M_2)_2 \cap E(C_1) = \emptyset$. Then $M_3 \cap E(C_1)$ is a 1-factor of C_1 . Let $M_1 = M'_1$, $M_2 = M'_2$, and $M'_3 =$

$(M_3 - E(C_1)) \cup (\mathcal{U} \cap E(C_1))$. Then M'_3 is a 1-factor of G and $|E(G) - \bigcup_{i=1}^3 M'_i| < |E(G) - \bigcup_{i=1}^3 M_i| = \mu_3(G)$, contradicting the minimality of $\mu_3(G)$.

Let $\text{girth}(G) < \mu_3(G)$. We first consider the case when G is trianglefree and $\mu_3(G) = 5$. If e is a bridge, then G contains a triangle, a contradiction. Hence we may assume that e is not a bridge.

It follows with Claim 2.3.1 that $|\bigcap_{i=1}^3 M_i| \leq 2$. If $|\bigcap_{i=1}^3 M_i| = 2$, then, by Theorem 2.3, G_c is a trianglefree graph on six vertices, eight edges, $\delta(G_c) = 2$, and four vertices of degree 3. The only realization of this constrains is a K_4 where two non-adjacent edges are subdivided. But the two bivalent vertices are incident to two edges of \mathcal{U} , and therefore $|\bigcap_{i=1}^3 M_i| > 2$, a contradiction. Thus we may assume that $\bigcap_{i=1}^3 M_i = \{e\}$. Hence G_c is a subdivision of K_2^3 and it follows that it contains a circuit of length 4 whose edges are in \mathcal{M} and \mathcal{U} , alternately. As in (***) we deduce a contradiction to the minimality of $\mu_3(G)$. Thus, G_c is cyclic and the statement for trianglefree graphs is proved.

Now let $\mu_3(G) \in \{4, 5, 6\}$. Suppose to the contrary that G_c contains a bridge. If G_c contains two bridges, then $\mu_3(G) = 6$ and the two bridges are connected by two parallel edges. Hence, they are bridges in G , a contradiction.

Thus we may assume that G_c contains precisely one bridge e . Let $e = xy$. Since e is incident to four edges of \mathcal{U} , it follows that G_c is connected. (Since for otherwise the other component is a circuit of length 4, and we obtain a contradiction as in (**).) Now there are two disjoint circuits C_x and C_y in G_c with $x \in V(C_x)$, and $y \in V(C_y)$. Theorem 2.3 implies that the $|E(C_x)| + |E(C_y)| \leq 10$. Furthermore $|\partial(V(G_c))| \leq 8$.

(***) Consider $(\bigcup_{i=1}^3 M_i) - E(G_c)$ as proper 3-edge-coloring ϕ of $G - E(G_c)$. In any case ϕ can be extended to a proper 4-edge-coloring of G which has a color class that contains precisely two edges. Now the result follows with Proposition 2.8 and Theorem 2.7.

Now suppose that no edge of $\bigcap_{i=1}^3 M_i$ is a bridge. If $|\bigcap_{i=1}^3 M_i| = 3$, then it follows with Theorem 2.3 that $\mu_3(G) = 6$ and G_c is a bridgeless cubic graph on six vertices. Hence $G_c = G$, a contradiction.

If $|\bigcap_{i=1}^3 M_i| = 2$, then $G_c - \bigcap_{i=1}^3 M_i$ is a cycle of total length $2\mu_3(G) - 4$. If it consists of more than one circuit, then $\mu_3(G) = 6$ and G_c contains a circuit of length 4 with two edges of \mathcal{M} and two of \mathcal{U} . We deduce a contradiction as in (**). Hence, $G_c - \bigcap_{i=1}^3 M_i$ is a circuit of length $2\mu_3(G) - 4$, and the edges of $\bigcap_{i=1}^3 M_i$ are chords of this circuit. We have $|\partial(V(G_c))| \leq 2\mu_3(G) - 8 \leq 4$. If a vertex v of G_c is adjacent to an edge of $M_i \cap \partial(V(G_c))$, then it has a neighbor in G_c which is also adjacent to an edge of $M_i - (M_j \cup M_k)$, $\{i, j, k\} = \{1, 2, 3\}$. Now we argue as in (***) to deduce that G is either 3-edge-colorable or it has oddness at most 2. Therefore it has a cyclic core by Theorem 2.7.

If $|\bigcap_{i=1}^3 M_i| = 1$, then $G_c - \bigcap_{i=1}^3 M_i$ is a cycle of total length $2\mu_3(G) - 2$. If it consists of more than one circuit, then $\mu_3(G) \in \{5, 6\}$ and G_c contains a circuit of length 4, where two edges of \mathcal{M} and two of \mathcal{U} . We deduce a contradiction as in (**). Hence $G_c - \bigcap_{i=1}^3 M_i$ is a circuit of length $2\mu_3(G) - 2$, and the edge e of

$\bigcap_{i=1}^3 M_i$ is a chord of this circuit. We have $|\partial(V(G_c))| \leq 2\mu_3(G) - 4 \leq 8$. Since there is an $i \in \{1, 2, 3\}$ such that $M_i \cap \partial(V(G_c)) \leq 2$, we deduce as in the case above that G has oddness at most 2 and hence it has a cyclic core by Theorem 2.7 \square

2.2 Berge-Fulkerson Conjecture

Following [17] we define a p -tuple edge multicoloring ($p > 1$) of a bridgeless cubic graph G as a list of $3p$ 1-factors (with possible repetitions) such that every edge is contained in precisely p 1-factors.

Theorem 2.10 ([17]) *Let G be a bridgeless cubic graph which has no non-trivial 3-edge-cut. If M is a 1-factor of G , then there are an integer $p > 1$ and a p -tuple edge multicoloring of G using M .*

Lemma 2.11 *Let G be a bridgeless cubic graph which has no non-trivial 3-edge-cut, M a 1-factor of G and P a path of length 3. If M contains no edge of P , then there is a 1-factor M' of G that contains the two endedges of P .*

Proof. Let P be a path with vertex set $\{v_1, \dots, v_4\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq 3\}$, and let $e = v_1 v_2$ and $e' = v_3 v_4$. We will show that there is a 1-factor M' that contains e and e' .

Let f_2, f_3 be the edges which are adjacent to v_2, v_3 , respectively, and which are no edges of P . If $f_2 = f_3$, (that is, v_2 and v_3 are connected by two edges) then every 1-factor that contains e has to contain e' .

Let $f_2 \neq f_3$, and $f_2, f_3 \in M$. Theorem 2.10 implies that there exist an integer $p > 1$ and a p -tuple edge multicoloring ϕ of G using M . Let M_1, \dots, M_p be the p 1-factors of ϕ that contain e . If M_i does not contain e' , then it contains f_3 . Since $f_3 \in M$ and $M_i \neq M$, for all $i \in \{1, \dots, p\}$, there is $j \in \{1, \dots, p\}$ such that $f_3 \notin M_j$. Hence, M_j contains e and e' . \square

Theorem 2.12 *Let G be a bridgeless cubic graph which has no non-trivial 3-edge-cut. If $\mu_3(G) \leq 4$, then G has a Fulkerson coloring.*

Proof. If $\mu_3(G) = 0$, then G is 3-edge-colorable, and it has a Fulkerson coloring. Let M_1, M_2, M_3 be three 1-factors of G such that $\mu_3(G) = |E(G) - \bigcup_{i=1}^3 M_i|$, and G_c be the induced core. Then $\mu_3(G) \geq 3$, by Corollary 2.4. Since G is trianglefree, it follows with Theorem 2.9 that G_c is cyclic. It is easy to see that the core is connected. Hence the edges of \mathcal{U} can be paired into at most two pairs (one pair and a single edge when $\mu_3(G) = 3$), such that the edges of a pair are connected by an edge of \mathcal{M} . Lemma 2.11 implies that there are two 1-factors M_4 and M_5 such that $\bigcup_{i=1}^5 M_i = E(G)$. Hence G has a Fulkerson coloring. \square

2.3 Short cycle covers

Theorem 2.13 *Let k, l, t be non-negative integers, and G be a cubic graph. If G has a k -core which has a l -cycle cover \mathcal{C}_c of length t , then G has a $(l+2)$ -cycle cover \mathcal{C} of length at most $\frac{4}{3}(|E(G)| - k) + t$. Furthermore, if \mathcal{C}_c is even, then \mathcal{C} is even.*

Proof. Let G_c be a k -core of G with respect to three 1-factors M_1, M_2, M_3 . For $i \neq j$, let $(M_i \cap M_j)_2 = (M_i \cap M_j) - \bigcap_{i=1}^3 M_i$, and $\mathcal{M}_2 = \bigcup_{i \neq j} (M_i \cap M_j)_2$. We have $|\mathcal{M}| = |\mathcal{M}_2| + |\bigcap_{i=1}^3 M_i|$, and with Claim 2.3.1 it follows that $k = |\mathcal{M}_2| + 2|\bigcap_{i=1}^3 M_i|$. Without loss of generality we assume that $|(M_1 \cap M_2)_2| + |(M_1 \cap M_3)_2| \geq \frac{2}{3}|\mathcal{M}_2|$.

Let \mathcal{C}_c be a l -cycle cover of G_c . For $i \in \{2, 3\}$, let $\mathcal{C}_i = (M_1 \cup M_i) - (M_1 \cap M_i)$, and $\mathcal{C} = \mathcal{C}_c \cup \{\mathcal{C}_2, \mathcal{C}_3\}$. Since \mathcal{C}_c covers $\mathcal{M} \cup \mathcal{U}$ and $\mathcal{C}_2 \cup \mathcal{C}_3$ covers $E(G) - (\mathcal{U} \cup \bigcap_{i=1}^3 M_i)$, it follows that \mathcal{C} is a cycle cover of G . Moreover, \mathcal{C}_i is an even cycle of length t_i , where $t_i = |M_1| + |M_i| - 2|M_1 \cap M_i| = |M_1| + |M_i| - 2(|(M_1 \cap M_i)_2| + |\bigcap_{i=1}^3 M_i|)$. Hence, $t_2 + t_3 = \frac{4}{3}|E(G)| - 2(|(M_1 \cap M_2)_2| + |(M_1 \cap M_3)_2| + 2|\bigcap_{i=1}^3 M_i|) \leq \frac{4}{3}|E(G)| - \frac{2}{3}(2|\mathcal{M}_2| + 6|\bigcap_{i=1}^3 M_i|) = \frac{4}{3}|E(G)| - \frac{2}{3}(2k + 2|\bigcap_{i=1}^3 M_i|) = \frac{4}{3}(|E(G)| - k) - |\bigcap_{i=1}^3 M_i| \leq \frac{4}{3}(|E(G)| - k)$.

Thus, \mathcal{C} is a $(l+2)$ -cycle cover of length $\frac{4}{3}(|E(G)| - k) + t$. Moreover, if \mathcal{C}_c is even, then \mathcal{C} is an even cycle cover of G . \square

The following result of Alon and Tarsi [1] and Bermond, Jackson and Jaeger [2] is the best known general result on the length of cycle covers.

Theorem 2.14 ([1, 2]) *Every bridgeless graph G has a 3-cycle cover of length at most $\frac{5}{3}|E(G)|$.*

Theorem 2.15 *Let $k \geq 0$, and G be a cubic graph. If G has a bridgeless k -core, then G has a cycle cover of length at most $\frac{4}{3}|E(G)| + 2k$.*

Proof. Let G_c be a bridgeless k -core of G with respect to three 1-factors M_1, M_2, M_3 . By Theorem 2.14, G_c has a cycle cover of length at most $\frac{5}{3}|E(G_c)|$. By Theorem 2.3 we have $|E(G_c)| = 2k - |\bigcap_{i=1}^3 M_i|$, and hence it follows with Theorem 2.13 that G has a cycle cover of length at most $\frac{4}{3}|E(G)| + 2k$. \square

We are going to prove better bounds for the length of cycle covers of a cubic graphs which have a bipartite core. First we show that bipartite cores are bridgeless.

Theorem 2.16 *Let G_c be a core of a cubic graph. If G_c is bipartite, then G_c is bridgeless.*

Proof. Let G_c be a core of G with respect to three 1-factors M_1, M_2, M_3 . If G_c is not bridgeless, then it has a component K_c that contains a bridge. Furthermore, K_c has a bridge e such that one component of $K_c - e$ is 2-edge-connected. Let K'_c

be a 2-edge-connected component of $K_c - e$. Since $e \in \bigcap_{i=1}^3 M_i$ and, by Theorem 2.2, $\mathcal{M} \cap E(K_c)$ is a 1-factor of K_c , it follows that $|V(K'_c)|$ is odd. Furthermore, if we remove all edges of $\bigcap_{i=1}^3 M_i$ from K'_c , then we obtain a set of circuits. Hence, G_c contains an odd circuit. Therefore, it is not bipartite. \square

Theorem 2.17 *Let $k \geq 0$, and G be a cubic graph. If G has a bipartite k -core G_c , then G has an even 4-cycle cover of length at most $\frac{4}{3}|E(G)| + \frac{2}{3}k$. In particular, if G_c is cyclic, then G has an even 3-cycle cover of length at most $\frac{4}{3}|E(G)| + \frac{2}{3}k$.*

Proof. Let G_c be a bipartite k -core of G with respect to three 1-factors M_1, M_2, M_3 . Let K_c be a component of G_c , and $k' = |E(K_c) \cap \mathcal{U}|$. It suffices to prove that every component K_c has an even cycle cover of length $2k'$. Then it follows that G_c has an even cycle cover of length $2k$. Hence G has an even cycle cover of length at most $\frac{4}{3}|E(G)| + \frac{2}{3}k$ by Theorem 2.13.

Clearly, if K_c is a circuit, then it has a 1-cycle cover of length $2k'$. Thus, the statement is true for G_c is cyclic.

If K_c is a component of G_c which is not a circuit, then, by Theorem 2.16, it is 2-edge-connected and $E(K_c) \cap \bigcap_{i=1}^3 M_i \neq \emptyset$. Furthermore K_c is a subdivision of a cubic multigraph H_c . Let $E^* = E(K_c) \cap \bigcap_{i=1}^3 M_i$. Since the two vertices which are incident to an edge of E^* have degree 3 in K_c , it follows that $K_c - E^*$ is a cycle. Since K_c is bipartite, it is an even cycle which has a proper 2-edge-coloring. Thus E^* is a color class of a proper 3-edge-coloring ϕ of K_c . Let P be a path between two trivalent vertices of K_c and all internal vertices of P have degree 2 in K_c . Since \mathcal{M} is a 1-factor of G_c , it follows that the endedges of P receive the same color. Hence, ϕ induces a proper 3-edge-coloring on H_c , where E^* is a color class. Let \mathcal{C}_{H_c} be a canonical cycle cover of H_c (of length $\frac{4}{3}|E(H_c)|$), which uses the edges of E^* twice. Now \mathcal{C}_{H_c} induces a 2-cycle cover \mathcal{C}_{K_c} of K_c . The length of \mathcal{C}_{K_c} is $|E(K_c)| + |E^*|$. By Theorem 2.3 we have $|E(K_c)| = 2k' - |E^*|$. Hence the length of \mathcal{C}_{K_c} is $2k'$.

It remains to show that \mathcal{C}_{K_c} is an even cycle. Every edge of E^* is contained in precisely two cycles of \mathcal{C}_{K_c} and all the other in precisely one. Let v be a vertex which is incident to two edges of \mathcal{U} . Then $d_{G_c}(v) = 3$, and v is incident to an edge of E^* . Hence every circuit C of \mathcal{C}_{K_c} does not contain any two consecutive edges of \mathcal{U} . Since \mathcal{M} is a 1-factor of G_c , it follows that the edges of C are in \mathcal{M} and \mathcal{U} alternately. Hence C has even length, and \mathcal{C}_{K_c} is an even 2-cycle cover of K_c . \square

Corollary 2.18 *Let G be a trianglefree bridgeless cubic graph. If $\mu_3(G) \leq 5$, then G has an even 3-cycle cover of length at most $\frac{4}{3}|E(G)| + 2$.*

Proof. If $\mu_3(G) \leq 5$, then Theorem 2.9 implies that the induced core is cyclic. If $\mu_3(G) = 5$, then $\frac{4}{3}|E(G)| + \lfloor \frac{2}{3}\mu_3(G) \rfloor$ is odd. Hence the result follows with Theorem 2.17. \square

In [5] it is proved that if a cubic graph G has a cyclic core, then it has a 3-cycle cover of length at most $\frac{14}{9}|E(G)|$. This result is improved to smaller than $\frac{14}{9}|E(G)|$ in [13]. We additionally deduce that there is an even 3-cycle cover of length smaller than $\frac{14}{9}|E(G)|$.

Corollary 2.19 *Let G be a cubic graph. If G has a cyclic core, then G has an even 3-cycle cover of length smaller than $\frac{14}{9}|E(G)|$.*

Proof. If G is 3-edge-colorable, then the statement is true. Let G_c be a cyclic k -core of G . Then G_c is 2-regular and it has $2k$ edges. Hence $2k \leq \frac{2}{3}|E(G)|$, and therefore $k \leq \frac{1}{3}|E(G)|$. If $k = \frac{1}{3}|E(G)|$, then G_c is an even 2-factor of G and hence, G is 3-edge-colorable. Thus, $k < \frac{1}{3}|E(G)|$ and Theorem 2.17 implies that G has an even 3-cycle cover of length smaller than $\frac{14}{9}|E(G)|$. \square

Kaiser, Král and Norine studied 1-factor covers of cubic graphs in [11]. We state their result in our notation.

Theorem 2.20 ([11]) *If G is a bridgeless cubic graph, then $\mu_3(G) \leq \frac{8}{35}|E(G)|$.*

Corollary 2.21 *Let $k \geq 0$, and G be a cubic graph with $\mu_3(G) = k$. If $\text{girth}(G) \geq k$ ($> k$), then G has an even 4-cycle cover (3-cycle cover) of length at most $\frac{52}{35}|E(G)|$.*

Proof. Let G_c be a k -core of G . Theorem 2.20 implies that $k \leq \frac{8}{35}|E(G)|$. If $\text{girth}(G) \geq k$ ($> k$), then Theorem 2.6 implies that G_c is bipartite (cyclic). We have $\frac{4}{3}|E(G)| + \frac{2}{3}k \leq \frac{52}{35}|E(G)|$ and hence, the statements follow with Theorem 2.17. \square

In [5] it is proved that if a cubic graph G has six 1-factors such that every edge is in precisely two of them, then it has a 3-cycle cover of length at most $\frac{22}{15}|E(G)|$. This bound is best possible for 3-cycle covers of bridgeless cubic graphs, since it is attained by the Petersen graph [13].

Corollary 2.22 *Let G be a cubic graph which has six 1-factors such that every edge is in precisely two of them.*

- 1) *Then G has an even 3-cycle cover of length at most $\frac{22}{15}|E(G)|$.*
- 2) *If $|V(G)| \not\equiv 0 \pmod{10}$, then G has an even 3-cycle cover of length smaller than $\frac{22}{15}|E(G)|$.*

Proof. 1) Let M_1, \dots, M_6 be a cover of G with six 1-factors such that every edge is contained in precisely two of them. Since $\binom{6}{2} = 15$, there are two 1 factors, say M_1, M_2 , such that $|M_1 \cap M_2| \leq \frac{1}{15}|E(G)|$. We claim that there is $i \in \{3, \dots, 6\}$ such that $|M_1 \cap M_2| + |M_1 \cap M_i| + |M_2 \cap M_i| \leq \frac{1}{5}|E(G)|$. Suppose to the contrary that this is not true. Then $\sum_{i=3}^6 (|M_1 \cap M_2| + |M_1 \cap M_i| + |M_2 \cap M_i|) > \frac{4}{5}|E(G)|$. We have $\sum_{i=3}^6 (|M_1 \cap M_2| + |M_1 \cap M_i| + |M_2 \cap M_i|) = \frac{2}{3}|E(G)| + 2|M_1 \cap M_2|$ and hence, $|M_1 \cap M_2| > \frac{1}{15}|E(G)|$, which contradicts our choice of M_1 and M_2 .

Let $i = 3$ and $|M_1 \cap M_2| + |M_1 \cap M_3| + |M_2 \cap M_3| \leq \frac{1}{5}|E(G)|$. Since every edge is contained in precisely two 1-factors, the k -core with respect to M_4 , M_5 and M_6 is cyclic and $k \leq \frac{1}{5}|E(G)|$. Theorem 2.17 implies that G has an even 3-cycle cover of length at most $\frac{22}{15}|E(G)|$.

2) If $|V(G)| \not\equiv 0 \pmod{10}$, then $|E(G)| \not\equiv 0 \pmod{15}$, and we deduce as above that G has cyclic k -core and $k < \frac{1}{5}|E(G)|$. Then the statement follows with Theorem 2.17 \square

Let G be a cubic graph which has a core. Clearly, if G has a bridge, then every core of G has a bridge. We conjecture that the opposite direction of that statement is true as well and propose two conjectures.

Conjecture 2.23 *Every bridgeless cubic graph has a bridgeless core.*

Conjecture 2.24 *Every bridgeless cubic graph has a bipartite core.*

Conjecture 1.2 implies Conjecture 2.24, which implies Conjecture 2.23, by Theorem 2.16.

3 4-covers

Let \mathcal{C} be a cycle cover of a graph G . For $e \in E(G)$, let $ced_{\mathcal{C}}(e) = |\{C : C \in \mathcal{C} \text{ and } e \in E(C)\}|$, and $\max\{ced_{\mathcal{C}}(e) : e \in E(G)\}$ be the *edge-depth* of G , which is denoted by $ced_{\mathcal{C}}(G)$. Zhang conjectured that every bridgeless graph has a shortest cycle cover of at most four cycles (Conjecture 8.11.5 in [19]), and that every 3-edge-connected graph has a shortest cycle cover \mathcal{C} such that $ced_{\mathcal{C}}(G) \leq 2$ (Conjecture 8.11.6 in [19]). The next theorem shows that we get the optimal bound for the length of a cycle cover if $\mu_4(G) = 0$, and that these graphs have a 5-cycle double cover. It also shows that these graphs satisfy the aforementioned conjectures of Zhang as well.

Theorem 3.1 *Let G be a cubic graph. If $\mu_4(G) = 0$, then*

- 1) *G has an even 4-cycle cover \mathcal{C} of length $\frac{4}{3}|E(G)|$, and $ced_{\mathcal{C}}(G) \leq 2$.*
- 2) *G has a 5-cycle double cover.*

Proof. Let $\mu_4(G) = 0$, and M^* be the set of four 1-factors M_1, \dots, M_4 of G with $E(G) = \bigcup_{i=1}^4 M_i$. Every edge is contained in at most two 1-factors and $\bigcap_{i=1}^4 M_i = \emptyset$. Every vertex is incident to precisely two edges which are in precisely one element of M^* , and to precisely one edge, which is in two elements of M^* . Thus the edges which are in precisely one 1-factor induce a 2-factor \mathcal{C}' on G , and the edges which are in two elements of M^* form a 1-factor M of G . For $t \in \{1, 2, 3\}$ and $N \subseteq E(G)$, let N^t (\bar{N}^t) be the set of edges of N ($E(G) - N$), which are in precisely t elements of M^* .

For $i \in \{1, \dots, 4\}$, let $\mathcal{C}_i = M_i^1 \cup \bar{M}_i^2$. Since G is not 3-edge-colorable, there is an edge $e \in M_i^1$. Since $M_i^1 = M_i - M$, it follows that \mathcal{C}_i consists of all circuits whose edges are in $M_i - M$ and M alternately. Hence every circuit of \mathcal{C}_i has even length.

(*) Every edge of M_i^1 is contained in precisely one cycle, namely in \mathcal{C}_i . Every edge e of M is contained in precisely two cycles, namely, if $e \in M_{i_1} \cap M_{i_2}$, then it is contained in \mathcal{C}_{i_3} and \mathcal{C}_{i_4} , where i_1, i_2, i_3, i_4 are pairwise different.

Let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$. Since $\bigcup_{i=1}^4 \mathcal{C}_i = E(G)$, it follows that \mathcal{C} is an even 4-cycle cover of G . Let t be the length of \mathcal{C} . Then (*) implies that $t = \sum_{i=1}^4 |M_i| = \frac{4}{3}|E(G)|$. Every edge of G is contained in at most 2 cycles. Hence $\text{ced}_{\mathcal{C}}(G) \leq 2$.

Since every edge of \mathcal{C}' is contained in precisely one cycle of \mathcal{C} , it follows that $\mathcal{C} \cup \mathcal{C}'$ is a 5-cycle double cover of G . \square

Theorem 3.1 2) was proved by Hou, Lai and Zhang [9] independently.

Following Zhang [19] we say that the Chinese postman problem is equivalent to the shortest cycle cover problem, if the shortest length of a closed trail that covers all edges of G is equal to the length of a shortest cycle cover. This is certainly true for cubic graphs G that have a cycle cover of length $\frac{4}{3}|E(G)|$. Zhang asked the following question (Problem 8.11.4 in [19]): Let $h \geq 5$ and G be a 3-edge-connected, cyclically h -edge-connected graph. If the Chinese Postman problem and the shortest cycle cover problem are equivalent for G , does G admit a nowhere-zero 4-flow? The answer to this question is negative since for $h \in \{5, 6\}$ there are cyclically h -edge connected snarks with $\mu_4(G) = 0$. It is known that $\mu_4(G) = 0$ if G is a flower snark or a Goldberg snark, see [6].

We now study the case, when the union of four 1-factors does not cover all edges of a cubic graph G .

Theorem 3.2 *Let G be a cubic graph that has four 1-factors M_1, \dots, M_4 with $|E(G) - \bigcup_{i=1}^4 M_i| = k \geq 0$. If $\bigcap_{i=1}^4 M_i = \emptyset$, then G has a 4-cycle cover of length $\frac{4}{3}|E(G)| + 4k$.*

Proof. If G is 3-edge-colorable, then the statement is true. Let G be not 3-edge-colorable. Since $\bigcap_{i=1}^4 M_i = \emptyset$, it follows that G is bridgeless.

Let M^* be the set of four 1-factors M_1, \dots, M_4 with $|E(G) - \bigcup_{i=1}^4 M_i| = k \geq 0$, and M be the set of edges which are in more than one element of M^* . For $t \in \{1, 2, 3\}$ and $N \subseteq E(G)$, let N^t (\bar{N}^t) be the set of edges of N ($E(G) - N$), which are in precisely t elements of M^* .

The case, when $k = 0$ is solved in Theorem 3.1. Let $k > 0$. For $i \in \{1, \dots, 4\}$, let $\mathcal{C}'_i = M_i^1 \cup \bar{M}_i^2 \cup M_i^3 \cup \mathcal{U}$. Every vertex of an edge of M_i^1 is incident either to an edge of \bar{M}_i^2 and to an edge of \bar{M}_i^1 , or to an edge of \bar{M}_i^3 and to an edge of \mathcal{U} . Every vertex of an edge of \bar{M}_i^2 is incident either to an edge of M_i^1 and to an edge of \bar{M}_i^1 , or to an edge of M_i^2 and to an edge of \mathcal{U} . Every vertex of an edge of M_i^3 is incident to an edge of \bar{M}_i^1 and an edge of \mathcal{U} . Every vertex of an edge of \mathcal{U} is incident either to an edge of M_i^1 and to an edge of \bar{M}_i^3 , or to an edge of M_i^2

and to an edge of \bar{M}_i^2 , or to an edge of M_i^3 and to an edge of \bar{M}_i^1 . Hence, every vertex of $G[\mathcal{C}'_i]$ is adjacent to precisely two edges of \mathcal{C}'_i ; that is, \mathcal{C}'_i is a cycle. Note that the edges of M_i^2 (and hence the vertices as well) are not in $G[\mathcal{C}'_i]$.

Thus $\bigcup_{i=1}^4 \mathcal{C}'_i = E(G)$. Let $\mathcal{C}' = \{\mathcal{C}'_1, \mathcal{C}'_2, \mathcal{C}'_3, \mathcal{C}'_4\}$. Then \mathcal{C}' is a 4-cycle cover of G . Each $e \in E(G)$ is either an element of \mathcal{U} or there are $i \in \{1, \dots, 4\}$ and $j \in \{1, 2, 3\}$ such that $e \in M_i^j$. If $e \in M_i^j$, then it is contained in precisely j cycles and if $e \in \mathcal{U}$, then it is contained in all four cycles. Hence the length of \mathcal{C}' is $\frac{4}{3}|E(G)| + 4k$. \square

Corollary 3.3 *Let G be a cubic graph. If $\mu_4(G) \in \{0, 1, 2, 3\}$, then G has an even 4-cycle cover of length $\frac{4}{3}|E(G)| + 4\mu_4(G)$.*

Proof. Let M_1, \dots, M_4 be four 1-factors of G with $\mu_4(G) = |E(G) - \bigcup_{i=1}^4 M_i|$. Since $\mu_4(G) \leq 3$, it follows that $\bigcap_{i=1}^4 M_i = \emptyset$. Therefore the result follows with Theorem 3.2. The bound is attained by the Petersen graph P with $\mu_4(P) = 1$. \square

4 Remark on hypohamiltonian snarks

A graph G is *hypohamiltonian* if it is not hamiltonian but $G - v$ is hamiltonian for every vertex $v \in V(G)$. Non 3-edge-colorable, cubic hypohamiltonian graphs are cyclically 4-edge-connected and have girth at least 5, and there are cyclically 6-edge-connected hypohamiltonian snarks with girth 6, see [15]. Since hamiltonian cubic graphs are 3-edge-colorable, and $G - v$ is not 3-edge-colorable for every snark G , hypohamiltonian snarks could be considered as being closest to being 3-edge-colorable. Indeed it is easy to see that if a cubic graph G has a vertex v such that $G - v$ is hamiltonian, then G has two 1-factors M_1, M_2 , such that $|M_1 \cap M_2| = 1$. Hence there is a third 1-factor M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Proposition 4.1 *If G is a hypohamiltonian snark, then G has a cyclic core.*

Corollary 2.4 implies, that if G is a cubic graph and $\mu_3(G) = 3$, then G has girth at most 6. It is easy to see that $\mu_3(G) = 3$, if G is the Petersen graph or a flower snark, which are hypohamiltonian snarks. Jaeger and Swart [10] conjectured that (1) the girth and (2) the cyclic connectivity of a snark is at most 6. The first conjecture is disproved by Kochol [12] and the second is still open. We believe that both statements of Jaeger and Swart are true for hypohamiltonian snarks.

Conjecture 4.2 *Let G be a snark. If G is hypohamiltonian, then $\mu_3(G) = 3$.*

Häggkvist [8] proposed to prove the Berge-Fulkerson conjecture for hypohamiltonian graphs, which might be easier to prove than the general case. By Theorem 2.12, Conjecture 4.2 implies that hypohamiltonian snarks have a Fulkerson coloring, and together with Theorem 2.17 it implies that they have an even 3-cycle cover of length at most $\frac{4}{3}|E(G)| + 2$.

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